MIDTERM - Modern Physics
Solutions - Part 1

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QUESTION 1 [20 points]

An object of mass 4m traveling at $\vec{v} = 0.9c\hat{x}$ collides head on with an object of mass 6m traveling at $\vec{v} = -0.6c\hat{x}$ to form a single object in the final state. [All masses are rest masses].

Part a)

Find the mass and velocity of this object. [15 points]

Here, we will use $m_1 = 4m$, $\vec{v_1} = 0.9c\hat{x}$, $m_2 = 6m$, and $\vec{v_2} = -0.6c\hat{x}$.

We first compute the Lorentz factors for each object.

$$\gamma_1 = \gamma_{0.9c} = \frac{1}{\sqrt{1-\beta_1^2}} = \frac{1}{\sqrt{1-0.9^2}} = 2.29 \tag{1}$$

$$\gamma_2 = \gamma_{0.6c} = \frac{1}{\sqrt{1-\beta_2^2}} = \frac{1}{\sqrt{1-0.6^2}} = 1.25 \tag{2}$$

The first principle we will use is **conservation of momentum**.

$$\gamma_f m_f \vec{v_f} = \gamma_1 m_1 \vec{v_1} + \gamma_2 m_2 \vec{v_2} \tag{3}$$

$$= (2.29)(4m)(0.9c\hat{x}) + (1.25)(6m)(-0.6c\hat{x}) \tag{4}$$

$$\gamma_f m_f v_f = 8.244mc - 4.5mc \qquad \text{Only in } \hat{x} \tag{5}$$

$$= 3.744mc \tag{6}$$

Using **conservation of energy**,

$$\gamma_f m_f c^2 = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2 \tag{7}$$

$$\gamma_f m_f = (2.29)(4m) + (1.25)(6m) \tag{8}$$

$$= 16.66m \tag{9}$$

Thus, dividing equation (6) by equation (9), we obtain:

$$\frac{3.744mc}{16.66m} = \frac{\gamma_f m_f v_f}{\gamma_f m_f} \tag{10}$$

Hence, we obtain (remembering we are dealing with the $\hat{x}$ direction):

$$v_f = 0.225c \tag{11}$$

The gamma factor associated with this speed is $\gamma_f = \frac{1}{\sqrt{1-0.225^2}} = 1.026$, such that we can plug those numbers back in equation (9) to obtain $m_f = 16.23m$.

Our final answer is thus (remember, we were asked for the velocity of the object, which is a vector):

$$\boxed{\vec{v_f} = 0.225c\hat{x}}$$

$$\boxed{m_f = 16.23m}$$

Part b)

Show that the invariant mass of the initial state equals the mass of the final state object. [5 points]

Here, we use the Lorentz invariant for the energy-momentum four-vector:

$$\left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 = m_{inv}^2 c^2 \tag{12}$$

Considering the initial state, where $\vec{p} = p_x\hat{x}$, we obtain:

$$
\begin{aligned}
m_{inv}^2c^2 &= \left(\frac{E_{init}}{c}\right)^2 - p_{x,init}^2 && (13)\\
&= [\gamma_1 m_1 c + \gamma_2 m_2 c]^2 - [\gamma_1 m_1 v_1 + \gamma_2 m_2 v_2]^2 && (14)\\
&= [(2.29)(4m)c + (1.25)(6m)c]^2 - [(2.29)(4m)(0.9c) + (1.25)(6m)(-0.6c)]^2 && (15)\\
&= [16.66mc]^2 - [3.744mc]^2 && (16)\\
m_{inv}^2c^2 &= 263.5m^2c^2 && (17)\\
m_{inv} &= 16.23m && (18)\\
&= m_f && (19)
\end{aligned}
$$

Thus, we obtain that the invariant mass of the initial state is equal to the final mass of the object, as computed in part (a).

QUESTION 2 [20 points]

Part a)

In frame S two events occur with the space-time coordinates $(x_1, y_1, z_1, t_1) = (a, 0, 0, \frac{a}{c})$***, and*** $(x_2, y_2, z_2, t_2) = (5a, 0, 0, \frac{a}{5c}$***. There is a reference frame S′ in which these two events are simultaneous (assume the origins of S and S′ coincide at*** $t = t' = 0$***). Find the velocity of the S′ frame with respect to frame S, as well as the time these events occur in S′. [10 points]***

Here, the key information is that the events should be simultaneous in S and S′. We will find the velocity v between the frames of reference (associated with a Lorentz factor γ and $\beta = \frac{v}{c}$). Here, all the motion is in the $\hat{x}$ direction, such that we will only worry about this one dimension. Using primed variables to refer to quantities measured in S′, this condition is written as:

$$t_1' = t_2' \tag{20}$$

$$\gamma \left(t_1 - \frac{\beta}{c} x_1 \right) = \gamma \left(t_2 - \frac{\beta}{c} x_2 \right) \tag{21}$$

$$t_1 - \frac{\beta}{c} x_1 = t_2 - \frac{\beta}{c} x_2 \tag{22}$$

$$\frac{a}{c} - \frac{\beta}{c} a = \frac{a}{5c} - \frac{\beta}{c} 5a \tag{23}$$

$$1 - \beta = \frac{1}{5} - 5\beta \tag{24}$$

$$4\beta = -\frac{4}{5} \tag{25}$$

$$\beta = -\frac{1}{5} \tag{26}$$

Hence, as we are asked for the velocity, $\vec{v} = -0.2c\hat{x}$, such that $\gamma = \frac{1}{\sqrt{1-0.2^2}} = 1.021$.

Plugging back to find t_1' and t_2',

$$t_1' = \gamma \left(t_1 - \frac{\beta}{c} x_1 \right) \tag{27}$$
$$= (1.021) \left(\frac{a}{c} - \frac{-0.2}{c} a \right) \tag{28}$$
$$= (1.021) \left(\frac{1.2a}{c} \right) \tag{29}$$
$$= 1.225 \frac{a}{c} = t_2' \tag{30}$$

You can verify that $t_1' = t_2'$.

Thus, our final answer is:

$$\boxed{\vec{v} = -0.2c\hat{x}}$$

$$\boxed{t_1' = t_2' = 1.225 \frac{a}{c}}$$

1 Q. 2 (b)

Q. Bob (at large negative x) runs toward the origin of reference frame S at speed 0.8c, while Anna (at large positive x) runs toward the origin at 0.6c. If Bob carries a pole of length $3m$ (in his rest frame) oriented in the direction in which he runs, what length would Anna measure for the rod in her reference frame? [10 points]

A. From the point of view of an observer in frame S Bob is running with velocity $u_B = 0.8c\,\hat{x}$ and Anna with velocity $v_A = -0.6c\,\hat{x}$. Bob carries a pole of length $3m$ and we want to know what length Anna would measure for the rod in her own reference frame. To do this we need to know what velocity Bob is travelling at according to Anna. This can be achieved through the relativistic addition of velocities formula:

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{31}$$

where $u'_x = u'_B$ is Bob's speed according to Anna, $u_x = u_B = 0.8c$ is Bob's speed according to an observer in the S frame and $v = v_A = -0.6c$ is Anna's speed according to an observer in the S frame. Plugging these into the equation above we get $u'_x = u'_B = 0.95c$.

Now we know that Anna sees Bob travelling toward her at $0.95c$ so we can work out the length of the rod she will measure using the length contraction formula $L = L_0/\gamma$. Using $v = 0.95c$ we get a gamma factor of 3.09. This results in a contracted length of $3m/3.09 = 0.97m$.

2 Q. 3

Q. Assume that a laser beam with 2mW of power and a wavelength of 311nm is incident on a metal photo-cathode. If the electrons emitted from the photo-cathode have a maximum velocity of 0.002c, answer the following questions:

(a) If the quantum efficiency of the photo-cathode is 75%, how many electrons per second leave the metal? [6 points]

A. A quantum efficiency of 75% means that 75% of the incoming photons result in outgoing electrons. Thus to calculate how many electrons leave we need to know how many photons are incident per second and multiply that number by 0.75. The energy of a single photon is:

$$E = \frac{hc}{\lambda} = \frac{hc}{311nm} = 4eV \tag{32}$$

The power of the laser is $2mW$ so the number of photons incident will be $2mW/4eV = 3.13 \times 10^{15}s^{-1}$. The number of electrons leaving per second is then $(3.13 \times 10^{15}s^{-1}) \times (0.75) = 2.35 \times 10^{15}e^{-}s^{-1}$.

(b) What is the maximum kinetic energy of the ejected electrons? [4 points]

A. We know the maximum velocity of the ejected electrons is $0.002c$ so we can use the relativistic formula for the KE to get KE_{max}:

$$KE_{max} = (\gamma - 1)\, m_e c^2 = 1.02eV \tag{33}$$

where $\gamma = \left(1 - 0.002^2\right)^{-1/2}$. Note that γ is very close to 1 as $v = 0.002c \ll c$. As a result we can treat the electrons non-relativistically and use the classical formula $KE = mv^2/2$ to get the same result.

(c) What is the work function of the metal? [6 points]

A. We know the energy of the incident photons from part (a) and the max KE of the outgoing electrons from part (b) so we can calculate the work function using the standard photoelectric effect relation:

$$\begin{aligned} KE_{max} &= hf - \phi \\ \Rightarrow \phi &= \frac{hc}{\lambda} - KE_{max} = 2.98eV \end{aligned} \tag{34}$$

(d) If the power of the laser were doubled, how would the answers to these questions change? [4 points]

A. (a) Double the power will mean double the number of incident photons. Thus the number of electrons leaving will also **double**.

(b) Doubling the power will not change the energy of the incident photons (this is determined by the wavelength) so the maximum kinetic energy of the outgoing electrons will **not change**. Remember that each photon can only liberate one electron.

(c) The work function is an intrinsic property of the material and will **not change** by changing parameters of the laser.