MAT186 Calculus I - Fall 2021

Midterm 2 Solutions — November 25, 2021

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1. (1 pt) Select the **ONE** equation, if any, that is **NOT TRUE**.

(a) $\ln(x^2+2x+1) - \ln(x+1) = \ln(x+1) - \ln(1)$
TRUE:

$$\begin{aligned}\ln(x^2+2x+1) - \ln(x+1) &= \ln\big(x+1\big)^2 - \ln(x+1)\\ &= 2\ln(x+1) - \ln(x+1) = \ln(x+1) - 0\\ &= \ln(x+1) - \ln(1)\end{aligned}$$

(b) $\ln(e^x 2^3) = x + 3\ln(2)$
TRUE:
$$\ln(e^x 2^3) = \ln(e^x) + \ln(2^3) = x + 3\ln(2).$$

(c) $e^{\ln(x+2)\ln(x)} = (x+2)^{\ln(x)}$
TRUE:
$$e^{\ln(x+2)\ln(x)} = (e^{\ln(x+2)})^{\ln(x)} = (x+2)^{\ln(x)}$$

(Notice also that $e^{\ln(x+2)\ln(x)} = (e^{\ln(x)})^{\ln(x+2)} = x^{\ln(x+2)}$)

(d) $2\ln(\frac{x}{x+1}) = (\ln(x) - \ln(x+1))^2$
FALSE:

$$\begin{aligned}2\ln(\frac{x}{x+1}) &= 2\big(\ln(x) - \ln(x+1)\big) = \ln(x^2) - \ln(x+1)^2\\ &\neq (\ln(x) - \ln(x+1))^2 = (\ln(x))^2 - 2\ln(x)\ln(x+1) + (\ln(x+1))^2\end{aligned}$$

(e) All of the above equations are true.

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2. (1 pt) The value of

$$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3}$$

is

(a) 0
(b) $\frac{1}{3}$
(c) $\frac{1}{6}$
(d) $+\infty$
(e) None of the above

We see by evaluation that the limit has the indeterminate form "$\frac{0}{0}$", so we can apply L'Hopital's rule directly to give

$$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} \widehat{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left(x - \tan^{-1} x\right)}{\frac{d}{dx}x^3} = \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2}$$
$$= \lim_{x\to 0} \frac{\frac{1+x^2-1}{1+x^2}}{3x^2} = \frac{1}{3}\lim_{x\to 0} \frac{x^2}{x^2(1+x^2)} = \frac{1}{3}\frac{1}{1+0} = \frac{1}{3}.$$

Hence the correct answer is (b).

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3. (1 pt) Select the **ONE** expression that is a solution to the following initial value problem.

$$\begin{cases} \frac{dy}{dt} = -(t+1)^2 y^2 e^t \\ y(0) = 1 \end{cases}$$

(a) $y(t) = 0$
(b) $y(t) = \frac{e^{-t}}{t^2+1}$
(c) $y(t) = \frac{e^{-t}}{t+1}$
(d) $y(t) = e^{-t/2}$
(e) $t = -1$

We can immediately eliminate (a.) as it does not satisfy the initial condition (even though a general solution to the differential equation) and (e.) as it is a value of time not a function of time. Notice also that (d.) satisfies the differential equation

$$\frac{dy}{dt} = -\frac{1}{2}e^{-t/2} = -\frac{y}{2}.$$

Hence the answer is either (b.) or (c.), notice both satisfy the initial condition $y(0) = 1$. So lets show if (b.) satisfies the differential equation directly using the product rule :

$$\frac{dy}{dt} = \frac{d}{dt}\Big(\frac{1}{t^2+1} \cdot e^{-t}\Big) = \frac{-2t}{(t^2+1)^2}e^{-t} - \frac{1}{t^2+1}e^{-t} = -\frac{t^2+1+2t}{(t^2+1)^2}e^{-t}$$
$$= -(t+1)^2\Big(\frac{e^{-t}}{(t^2+1)^2}\Big)e^{-t}e^t = -(t+1)^2 y^2 e^t.$$

Hence we see that the answer is (b.).

4. (1 pt) Consider the initial value problem

$$\frac{dy}{dt} = t + y$$
$$y(0) = 2.$$

Estimate $y(4)$ using Euler's method with a step-size $h = 1$.

(a) $y(4) \approx 19$

(b) $y(4) \approx 23$

(c) $y(4) \approx 43$

(d) $y(4) \approx 47$

(e) $y(4) \approx 90$

We shall use the Euler formula

$$y_{n+1} = y_n + (t_n + y_n) \cdot h,$$

where $y_0 = 2$ and $t_{n+1} = t_n + 1$ with $t_0 = 0$. We do this recursively in four steps to find $y_4 \approx y(4)$:

$$y_1 = 2 + (0 + 2) \cdot 1 = 4$$
$$y_2 = 4 + (1 + 4) \cdot 1 = 9$$
$$y_3 = 9 + (2 + 9) \cdot 1 = 20$$
$$y_4 = 20 + (3 + 20) \cdot 1 = 43.$$

Thus, $y(4) \approx 43$ that is the correct answer is (c.).

5. (1 pt) Fred decides to raise alpacas on his farm. Let $W(x)$ be the amount of wool, in kilograms per year, produced by an alpaca who was fed x kilograms of food per day. Select the **ONE** sentence that gives a valid interpretation of the statement $W'(4) = 0.7$.

(a) To increase an alpaca's wool production from **4 kilograms** per year to **4.6 kilograms** per year, Fred should feed it about **0.42 extra kilograms** of food per day.

(b) To increase a alpaca's wool production from **4 kilograms** per year to **4.1 kilograms** per year, Fred should feed it approximately **0.7 more kilograms** of food per day.

(c) An alpaca that is fed **4.3 kilograms** of food per day instead of **4 kilograms** of food per day will produce approximately **0.21 additional kilograms** of wool per year.

(d) An alpaca that is fed **4.1 kilograms** of food per day instead of **4 kilograms** of food per day will produce approximately **0.7 additional kilograms** of wool per year.

(e) An alpaca that is fed **4 kilograms** of food per day will produce **0.7 kilograms** of wool per year.

Options (a) and (b) are eliminated because 4 represents food and 0.7 represents wool, not the other way around. Option (e) is eliminated because it represents the statement $W(4) = 0.7$ about the original function, not the derivative. The derivative $W'(4)$ represents the instantaneous rate at which wool production would respond in changes to food delivery. This rate is 0.7 kg wool *per additional kg of food*. Since in option (c), we increase food by 0.3 kilograms, wool would increase approximately

$$0.3 \text{ kg food} \cdot \frac{0.7 \text{ kg wool}}{\text{kg food}} = 0.21 \text{ kg wool}.$$

Option (d) is incorrect because if we increased food by 0.1 kg, that would cause wool to increase by approximately $0.1 \cdot 0.7 = 0.07$ kg.

You can also think of this in terms of the linear approximation $\Delta W = W'(4)\Delta X$.

6. (1 pt) You were feeling adventurous and you agreed to go skydiving with your friend. After jumping from the plane, your speed (in metres) for the first 90 seconds was given by the following function:

$$f(t) = \begin{cases} 58(1 - e^{-t/3}), & 0 \le t \le 45 \qquad \text{(\textit{before} opening the parachute)} \\ (53 - 58e^{-15})e^{90-2t} + 5, & 45 < t \le 90 \qquad \text{(\textit{after} opening the parachute)} \end{cases}$$

Which approach, if any, gives an approximation that *guarantees* an **underestimate** of the distance that you traveled in the first 90 seconds?

(a) Using a left-endpoint Riemann sum for the interval $[0, 90]$

(b) Using a right-endpoint Riemann sum for the interval $[0, 90]$

(c) Using a left-endpoint Riemann sum for $t \in [0, 45]$ and a right-endpoint Riemann sum for $t \in [45, 90]$

(d) Using a right-endpoint Riemann sum for $t \in [0, 45]$ and a left-endpoint Riemann sum for $t \in [45, 90]$

(e) More information is required to guarantee an underestimate.

We recall that on an interval, the left-hand sum is an over-estimate if the function is strictly decreasing and an underestimate if the sum is strictly increasing. The opposite is true for the right-hand sum, namely overestimate if the function is strictly increasing and an underestimate if the function is strictly decreasing. We calculate the derivatives on the separate intervals,

$$f'(t) = +\frac{58}{3}e^{-t/3} > 0 \quad \text{for} \quad t \in [0, 45]$$

and

$$f'(t) = -2(53 - 58e^{-15})e^{90-2t} < 0 \quad \text{for} \quad t \in [45, 90]$$

Hence from the above dialogue concerning under/over-estimates, if we want an underestimate over $[0, 90]$ then we want to use a left-endpoint Riemann sum for $t \in [0, 45]$ and a right-endpoint Riemann sum for $t \in [45, 90]$. Thus the answer is (c).

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7. (1 pt) Which **ONE** of the following expressions is equal to the sum of cubes from 2 to 20: $\sum_{i=2}^{20} i^3$?

For each of these we can raise or lower the indices to see whether we can reproduce the desired sum.

(a) $\sum_{j=0}^{18} j^3$ (NO)

(b) $\sum_{k=-10}^{8} (k-12)^3 = \sum_{i=-10-12}^{8-12} ((i+12)-12)^3 = \sum_{i=-22}^{-4} i^3$ (NO)

(c) $\sum_{l=5}^{23} (l+3)^3 = \sum_{i=5+3}^{23+3} ((i-3)+3)^3 = \sum_{l=8}^{26} i^3$ (NO)

(d) $\sum_{m=1}^{19} (m+1)^3 = \sum_{i=1+1}^{19+1} ((i-1)+1)^3 = \sum_{l=2}^{20} i^3$ (YES)

(e) $\sum_{n=-2}^{18} (n+4)^3 = \sum_{i=-2+4}^{18+4} ((i-4)+4)^3 = \sum_{n=2}^{22} i^3$ (NO)

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8. (1 pt) Which of the following limits, if any, is equal to the definite integral : $\int_2^5 x^2\,dx$?

(a) $\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(2 + \frac{3(j+1)}{n}\right)^2 \left(\frac{3}{n}\right)$

(b) $\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(2 + \frac{5j}{n}\right)^2 \left(\frac{5}{n}\right)$

(c) $\lim_{n\to\infty} \sum_{j=2}^{5} (j)^2 \left(\frac{3}{n}\right)$

(d) $\lim_{n\to\infty} \sum_{j=2}^{n+1} \left(\frac{3j}{n}\right)^2 \left(\frac{3}{n}\right)$

(e) None of the above

We see that (c) definitely cannot be a limit of Riemann sums. Option (b) is also eliminated because the width Δx of the interval is wrong. Option (d) is also eliminated because the leftmost and rightmost ‘endpoints’ ($3j/n$ when $j = 2$ and when $j = n+1$) would not converge to 2 and 5. By reindexing, option (a) is indeed a correct limit of a right-endpoint Riemann sum. Hence the correct answer is (a).

END of Multiple Choice part

9. (5 pts total) A rectangle expands such that its length is always twice as long as its width. *(This statement applies to parts (a), (b), and (c).)*

(a) (1 pt) When its width is growing at 5 cm per minute, at what rate is its length increasing?

Let's call the width W and the length L. Then from the above statement we note that $L = 2W$. We are given $\frac{dW}{dt} = 5\frac{\text{cm}}{\text{min}}$ and we want to find $\frac{dL}{dt} = ?$. If we differentiate our relationship then we find

$$\frac{dL}{dt} = 2\frac{dW}{dt} = 10\frac{\text{cm}}{\text{min}}.$$

Length is increasing at $10\frac{\text{cm}}{\text{min}}$

(b) (2 pts) When its width is equal to 7 cm, and the width is growing at 5 cm per minute, at what rate is the area of this rectangle increasing? (You do not have to simplify your answer)

Here we are again given that $\frac{dW}{dt} = 5\frac{\text{cm}}{\text{min}}$ and we want to find $\frac{dA}{dt} = ?$ when $W = 7\text{cm}$. Recall that $L = 2W$ so $L = 14$ at this particular moment. As in part (a), differentiating $L = 2W$ with respect to time we get $\frac{dL}{dt} = 2\frac{dW}{dt} = 10$.
Differentiating the area $A = WL$ with respect to time, using the product rule and plugging in our known quantities gives

$$\frac{dA}{dt} = \frac{dW}{dt}L + W\frac{dL}{dt} = (5)(14) + (7)(10) = 140\frac{\text{cm}^2}{\text{min}}.$$

Area is increasing at $140\frac{\text{cm}^2}{\text{min}}$

It is also possible to use $L = 2W$ to rewrite $A = WL$ as either $A = 2W^2$ or $A = \frac{1}{2}L^2$. In these cases taking the derivative (and using the chain rule where required) yields

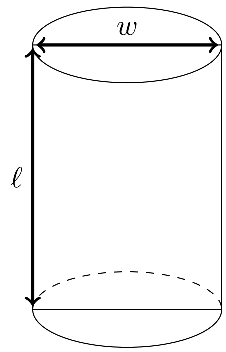
$$\frac{dA}{dt} = 4W\frac{dW}{dt} = (4)(7)(5) = 140$$

or

$$\frac{dA}{dt} = L\frac{dL}{dt} = (14)(10) = 140$$

(c) (2 pts) The rectangle- **still with its length always equal to twice its width**- is the middle cross-section of a cylinder, so that the length of the rectangle is the **height** of the cylinder and the width of the rectangle is the **diameter** of the circular end of the cylinder.

When the height of this cylinder is 20 cm and is increasing at 30 cm per minute, at what rate is its volume expanding? (You do not have to simplify your answer)



First, lets recall that a cylinder of height h and base radius r has a volume of

$$V = \pi r^2 h.$$

From the information given in the first paragraph notice that $h = L$ and $r = \dfrac{W}{2}$. We can rewrite the volume as $V = \frac{\pi}{4}W^2 L$. Differentiating with respect to time, and using the product and chain rules where required, gives the change in volume by

$$\frac{dV}{dt} = \frac{\pi}{2}W\frac{dW}{dt}L + \frac{\pi}{4}W^2\frac{dL}{dt}$$

We are given that $L = 20\text{ cm}$ and $\frac{dL}{dt} = 30\frac{\text{cm}}{\text{min}}$. Again from $L = 2W$ and $\frac{dL}{dt} = 2\frac{dW}{dt}$ we find that $W = 10$ and $\frac{dW}{dt} = 15$. We substitute in the known quantities at this instant.

$$\frac{dV}{dt} = \frac{\pi}{2}(10)(15)(20) + \frac{\pi}{4}(10)^2(30) = 2,250\pi\frac{\text{cm}^3}{\text{min}}.$$

Volume is expanding at $2,250\pi\frac{\text{cm}^3}{\text{min}}$

We could write the volume as a function of L :

$$V = \pi(\frac{L}{4})^2 L = \frac{\pi}{16}L^3.$$

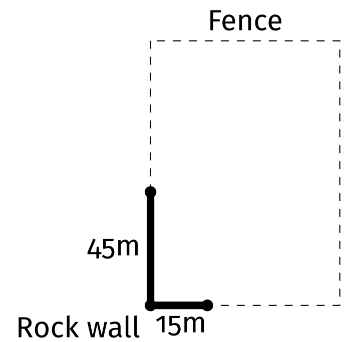
Differentiating and substituting known quantities at the instant would yield

$$\frac{dV}{dt} = \frac{3\pi}{16}L^2\frac{dL}{dt} = \frac{3\pi}{16}(20)^2(30) = 2,250$$

Something similar can be done with W, giving the same result.

10. (4 pts total) Fred is building a rectangular pen[1] for his alpacas[2]. He will use the an existing rock wall as part of his pen. The existing rock wall consists of a 15 meter segment forming a corner with a 45 meter segment. The **entire** rock wall must be used as part of the pen.

(a) (2 pts) If Fred has 140 meters of wooden fencing to add, what is the area of the largest pen he can build? *(the drawing does not necessarily reflect what the optimal case looks like)*



We first define the horizontal part of our rectangular pen by x and the vertical part of our pen by y. Then the area of our rectangular pen is given by $A = x \cdot y$. We will now take note that we have a constraint on the perimeter $P = 140\text{m}$, which is written in term of x and y as

$$140 = P = x + (x - 15) + y + (y - 45) \implies x + y = 100.$$

Hence we want to optimize the area $A = x \cdot y$ subject to the constraint $x + y = 100$. We now substitute our constraint into the area function so that $A = A(x)$ and we want to maximize the function

$$A(x) = x \cdot (100 - x) = 100x - x^2 \qquad 15 \leq x \leq 55.$$

Notice the range in $x \in [15, 55]$ is required since the **entire** rock wall must be used. Clearly x must be at least as large as the horizontal part of the wall, 15m. To see why x can't be bigger than 55m, substitute in the smallest possible y-value, 45m, into the constraint equation $x + y - 100$.

Then we check for critical points by taking the derivative

$$A'(x) = 100 - 2x = 0 \implies x = 50 \in [15, 55].$$

Therefore the global maximum on this interval could be at $x = 50$ or at one of the endpoints $x = 15$ or $x = 55$. Notice that $A'(x) = 100 - 2x > 0$ when $x < 50$ and $A'(x) = 100 - 2x < 0$ when $x > 50$. Hence by the first derivative test this point is indeed a local maximum. Moreover, the function increases from $x = 15$ to $x = 50$, and decreases from $x = 50$ to $x = 55$. Thus, the area is maximized when $x = y = 50$ with an area of

$$A(20) = 50 \cdot 50 = 2,500.$$

[1] A pen is a fenced-in area for animals.

[2] An alpaca is a cute and fuzzy animal.

Area of the pen: 2500m^2

You could also compare the critical point the endpoints ($A(15)$, $A(50)$, and $A(55)$) to see which has the largest value, but this requires more multiplication by hand.

(b) (2 pts) If instead Fred only has 110 meters of wooden fencing to add, what is the area of the largest pen he can build?

The setup is the same, albeit with different constraint on the perimeter $P = 110\text{m}$, which is written in term of x and y as

$$110 = P = x + (x - 15) + y + (y - 45) \implies x + y = 85.$$

Hence we want to optimize the area $A = x \cdot y$ subject to the constraint $x + y = 85$. We now substitute our constraint into the area function so that $A = A(x)$ and we want to maximize the function

$$A(x) = x \cdot (85 - x) = 85x - x^2 \qquad 15 \leq x \leq 40.$$

We check for critical points by taking the derivative

$$A'(x) = 85 - 2x = 0 \implies x = 85/2.$$

Notice that $x = 85/2 = 42.5 > 40$, which is outside our interval. Hence we compare the endpoints

$$A(15) = 15 \cdot 60 = 10 \cdot 60 + 5 \cdot 60 = 600 + 300 = 900.$$
$$A(40) = 40 \cdot 45 = 20 \cdot 90 = 1800.$$

Hence by the extreme value theorem, the max area is given by 1800 when $x = 40$ and $y = 45$.

Area of the pen:1800m^2

11. (5 pts total) Lego Batman has been trapped by Scarecrow. Scarecrow has created a new liquid called Globbo that behaves like water but stays in a perfect sphere even under gravity.

Batman has been thrown into a sphere of Globbo. The Globbo sphere is evaporating at a rate proportional to its surface area (with constant of proportionality α) and Batman wants to know how long it will take for it to get small enough to manage the situation.

(a) (3 pts) At first, Lego Batman decides to figure out how quickly the mass of the liquid is shrinking. He knows that the density of Globbo is 2,000 kg/m^3 and that the sphere had a radius of 3 m when he was thrown into it. Find a differential equation and initial condition for the **mass** that uses only the following unknowns: mass, time, and the constant of proportionality.

Let the density be given by $\rho = 2,000\frac{\text{kg}}{\text{m}^3}$ and we recall that the volume of a sphere $V = \frac{4\pi}{3}R^3$ where R is the radius of a sphere. The mass $M = M(t)$ is related to these quantities via

$$M = \rho V = \frac{4\rho\pi}{3}R^3\,\frac{\text{kg}}{\text{m}^3} \implies R = \left(\frac{3M}{4\rho\pi}\right)^{1/3}.$$

Recall that the surface area of a sphere is $A = 4\pi R^2$. Since the volume is changing in-time proportional(with constant α) to the surface area A, we have $\frac{dV}{dt} = \alpha A$. Then, after differentiating $M = \rho V$ in-time and substituting, we have

$$\frac{dM}{dt} = \rho\frac{dV}{dt} = \rho\alpha A = 4\rho\pi\alpha R^2 = (\rho)(\alpha)(4\pi\left(\frac{3M}{4\rho\pi}\right)^{2/3})$$

From our original relationship between the radius R and mass M, we see that the initial mass $M(0)$ at $R = 3\text{ m}$ satisfies

$$M(0) = \frac{4\rho\pi}{3}(3\text{ m})^3\,\frac{\text{kg}}{\text{m}^3} = 36\rho\pi\text{ kg}.$$

Differential equation: $\frac{dM}{dt} = (\rho)(\alpha)(4\pi\left(\frac{3M}{4\rho\pi}\right)^{2/3})$
Initial condition: $M(0) = 36\rho\pi\text{ kg}$

There are two other valid interpretations of the differential equation. One is that you could have assumed 'evaporation' means $\frac{dM}{dt} = \alpha A$ instead of $\frac{dV}{dt} = \alpha A$; in this case your solution would simply be missing the one factor of ρ out front. The other is that you could have assumed α had to be positive so you put $-\alpha$ to represent a decreasing volume.

(b) (2 pts) After getting the first result, Batman tries a new approach and wants to figure out the radius of the sphere. Find a differential equation for the **radius** that uses only the following unknowns: radius, time, and the constant of proportionality.

In order to find the differential equation for the Radius $R = R(t)$ we differentiate $V = \frac{4}{3}\pi R^3$ in terms of time.

$$\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$$

Then, since we are given $\frac{dV}{dt} = \alpha A = \alpha 4\pi R^2$, we can set these two expressions for $\frac{dV}{dt}$ equal to one another.

$$4\pi R^2 \frac{dR}{dt} = \alpha 4\pi R^2$$
$$\implies \frac{dR}{dt} = \alpha$$

Differential equation: $\frac{dR}{dt} = \alpha$

If you assumed $\frac{dM}{dt} = \alpha A$ instead of $\frac{dV}{dt} = \alpha A$, your solution would be $\frac{dR}{dt} = \alpha/\rho$. You may also have a negative sign in front of α.

END of Long Answer part